An explicit formula of powers of the 2 × 2 quantum matrices and its applications

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Abstract

We present an explicit formula of the powers for the 2×2 quantum matrices, that is a natural quantum analogue of the powers of the usual 2 × 2 matrices. As applications, we give some non-commutative relations of the entries of the powers for the 2 × 2 quantum matrices, which is a simple proof of the results of Vokos-Zumino-Wess (1990).

1 Introduction

Let A be a 2×2 matrix over a fixed base field k and $a,b,c,d \in k$ are entries of A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For any positive number n, the following explicit formula of A^n holds:

$$A^n = \begin{pmatrix} af_{n-1}(\mathrm{tr}\ A) - \det Af_{n-2}(\mathrm{tr}\ A) & bf_{n-1}(\mathrm{tr}\ A) \\ cf_{n-1}(\mathrm{tr}\ A) & df_{n-1}(\mathrm{tr}\ A) - \det Af_{n-2}(\mathrm{tr}\ A) \end{pmatrix} \tag{1.1}$$

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[1] Dedicated to T. Umeda and M. Wakayama for their 66th birthdays.

Here $f_n(x)$ is the polynomial of degree n defined by

$$f_n(x) = f_n(x,y) := \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^l \binom{n-l}{l} x^{n-2l} y^l, \quad f_{-1}(x) := 0 \quad , \qquad (1.2)$$

which is the Chebyshev polynomial of the second kind:

$$U_n(x) := \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^l \binom{n-l}{l} x^{n-2l}.$$

In fact this formula is well-known in linear algebra, and its proof is easy by induction and the recurrence relation for $f_n(x)$:

$$f_{n+1}(x) = xf_n(x) - yf_{n-1}(x). \qquad (1.3)$$

In this note, we give a quantum analogue of the formula (1.1) and its applications. First, we review some fundamental objects and facts on quantum matrix or groups [M], [T] relevant to the main results of this paper.

We call A a 2×2 (q-)quantum matrix if its entries satisfy the following relations:

$$\begin{aligned} &ab = qba, \quad ac = qca, \quad ad - da = (q - q^{-1})bc, \\ &bc = cb, \quad bd = qdb, \quad cd = qdc, \end{aligned} \qquad (R_q)$$

Here q is a central indeterminate. A quantum analogue of the coordinate ring $A_q(Mat(2))$ is the algebra generated by a,b,c,d and q which is a typical example of quantum groups.

The quantum adjoint matrix of any quantum matrix A is defined as

$$\hat{A} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} := \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}. \qquad (1.4)$$

By the definition of $\hat{A}$ and the relations (R_q), the quantum adjoint matrix $\hat{A}$ satisfies the relations ($R_{q^{-1}}$):

$$\begin{aligned} &\hat{a}\hat{b} = q^{-1}\hat{b}\hat{a}, \quad \hat{a}\hat{c} = q^{-1}\hat{c}\hat{a}, \quad \hat{a}\hat{d} - \hat{d}\hat{a} = (q^{-1} - q)\hat{b}\hat{c}, \\ &\hat{b}\hat{c} = \hat{c}\hat{b}, \quad \hat{b}\hat{d} = q^{-1}\hat{d}\hat{b}, \quad \hat{c}\hat{d} = q^{-1}\hat{d}\hat{c}. \end{aligned}$$

Hence, the relations (R_q) are equivalent to

$$bc = cb, \quad A\hat{A} = \hat{A}A = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} = \delta E_2, \tag{1.5}$$

where E_2 is the 2×2 identity matrix, and $\delta := ad - qbc = da - q^{-1}bc$ is the quantum determinant of A which is a central element of $\mathrm{A}_q(Mat(2))$.

For convenience, we introduce a 2 × 2 matrix

$$C := \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}$$

and put

$$\tau := \mathrm{tr}\ (AC) = q^{\frac{1}{2}}a + q^{-\frac{1}{2}}d, \quad \tau' := \mathrm{tr}\ (C^{-1}A) = q^{-\frac{1}{2}}a + q^{\frac{1}{2}}d.$$

Our main results are following.

Theorem 1.1. For any positive integer n, we have

$$A^n = AC^{-n+1}f_{n-1}(\tau) - C^{-n}\delta f_{n-2}(\tau) \tag{1.6}$$

$$= f_{n-1}(\tau')C^{n-1}A - f_{n-2}(\tau')\delta C^n, \tag{1.7}$$

where

$$f_n(\tau) := f_n(\tau, \delta) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^l \binom{n-l}{l} \tau^{n-2l}\delta^l, \quad f_{-1}(\tau) := 0.$$

Let us put

$$A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \quad \hat{A}^m := \begin{pmatrix} \hat{a}_m & \hat{b}_m \\ \hat{c}_m & \hat{d}_m \end{pmatrix}.$$

By comparing the entries of A^n and (1.6), (1.7), we obtain the following explicit formulas of the entries of A^n.

Corollary 1.2. For any positive integer n, we have

$$a_n = q^{-\frac{n-1}{2}} af_{n-1}(\tau) - q^{-\frac{n}{2}}\delta f_{n-2}(\tau) = q^{\frac{n-1}{2}} f_{n-1}(\tau')a - q^{\frac{n}{2}}\delta f_{n-2}(\tau'), \tag{1.8}$$

$$b_n = q^{\frac{n-1}{2}} b f_{n-1}(\tau) = q^{\frac{n-1}{2}} f_{n-1}(\tau') b, \tag{1.9}$$

$$c_n = q^{-\frac{n-1}{2}} c f_{n-1}(\tau) = q^{-\frac{n-1}{2}} f_{n-1}(\tau') c, \tag{1.10}$$

$$d_n = q^{\frac{n-1}{2}} d f_{n-1}(\tau) - q^{\frac{n}{2}} \delta f_{n-2}(\tau) = q^{-\frac{n-1}{2}} f_{n-1}(\tau') d - q^{-\frac{n}{2}} \delta f_{n-2}(\tau'). \tag{1.11}$$

As applications, we give the following results, in particular Theorem 1.4.

Corollary 1.3. For any positive integer m, we have

$$\begin{aligned} \hat{a}_m &= q^{\frac{m-1}{2}} a \hat{f}_{m-1}(\tau) - q^{\frac{m}{2}} \delta f_{m-2}(\tau) \\ &= q^{-\frac{m-1}{2}} f_{m-1}(\tau') \hat{a} - q^{-\frac{m}{2}} \delta f_{m-2}(\tau') = d_m, \end{aligned} \tag{1.12}$$

$$\hat{b}_m = q^{-\frac{m-1}{2}} \hat{b} f_{m-1}(\tau) = q^{-\frac{m-1}{2}} f_{m-1}(\tau') \hat{b} = -q^{-m} b_m, \tag{1.13}$$

$$\hat{c}_m = q^{\frac{m-1}{2}} c \hat{f}_{m-1}(\tau) = q^{\frac{m-1}{2}} f_{m-1}(\tau') \hat{c} = -q^m c_m, \tag{1.14}$$

$$\begin{aligned} \hat{d}_m &= q^{-\frac{m-1}{2}} d \hat{f}_{m-1}(\tau) - q^{-\frac{m}{2}} \delta f_{m-2}(\tau) \\ &= q^{-\frac{m-1}{2}} f_{m-1}(\tau') \hat{d} - q^{-\frac{m}{2}} \delta f_{m-2}(\tau') = a_m. \end{aligned} \tag{1.15}$$

Theorem 1.4 ([VZW])**.** For any non-negative integers m and n, we have

$$d_m a_n - q^{-m} b_m c_n = a_n d_m - q^m b_n c_m = \begin{cases} \delta^m a_{n-m} & (m < n) \\ \delta^n d_{m-n} & (m \ge n) \end{cases}, \tag{1.16}$$

$$d_m b_n - q^{-m} b_m d_n = -q^{-m} a_n b_m + b_n a_m = \begin{cases} \delta^m b_{n-m} & (m < n) \\ -q^{n-m} \delta^n b_{m-n} & (m \ge n) \end{cases}, \tag{1.17}$$

$$-q^m c_m a_n + a_m c_n = c_n d_m - q^m d_n c_m = \begin{cases} q^m \delta^m c_{n-m} & (m < n) \\ -q^{n-m} \delta^n c_{m-n} & (m \ge n) \end{cases}, \tag{1.18}$$

$$-q^{m} c_m b_n + a_m d_n = -q\, c_n b_m + d_n a_m = \begin{cases} {}^{-m}\delta^m d_{n-m} & (m < n) \\ \delta^n a_{m-n} & (m \geq n), \end{cases} \qquad (1.19)$$

$$b_n c_m - q^{n-m} c_n b_m = 0. \qquad (1.20)$$

2 Proof of Theorem 1.1

To prove Theorem 1.1, we need a quantum analogue of Cayley-Hamilton theorem.

Lemma 2.1 ([UW] Lemma 3)**.** The following formula holds.

$$A^2 = AC^{-1}\tau - C^{-2}\delta = \tau' CA - \delta C^2. \qquad (2.1)$$

Proof of Theorem 1.1 Since (1.6) and (1.7) can be similarly proved, we only prove (1.6). These formulas are proved by induction on n.

The $n = 1$ case is trivial. Assume the case of n holds. Hence, from the induction hypothesis we have

$$\begin{aligned} A^{n+1} &= AA^n \\ &= A\left\{AC^{-n+1} f_{n-1}(\tau) - C^{-n}\delta f_{n-2}(\tau)\right\} \\ &= A^2 C^{-n+1} f_{n-1}(\tau) - AC^{-n}\delta f_{n-2}(\tau). \end{aligned}$$

By Cayley-Hamilton theorem (2.1) and the recursion (1.3), we have

$$\begin{aligned} A^{n+1} &= (AC^{-1}\tau - C^{-2}\delta)C^{-n+1} f_{n-1}(\tau) - AC^{-n}\delta f_{n-2}(\tau) \\ &= AC^{-n}(\tau f_{n-1}(\tau) - \delta f_{n-2}(\tau)) - C^{-n-1}\delta f_{n-1}(\tau) = AC^{-n} f_n(\tau) \\ &\quad - C^{-n-1}\delta f_{n-1}(\tau). \end{aligned}$$

The formula (1.7) can be proved by the similar argument for $A^{n+1} = A^n A$.

□

Corollary 1.2 follows from comparing the entries of A^n and (1.6), (1.7) immediately.

Remark 2.2. (1) By consider the classical limit $q = 1$ in Corollary 1.2, we recover the classical result (1.1).
(2) From the recursion (1.3), we derive other expressions of a_n and d_n:

$$a_n = q^{-\frac{n}{2}} f_n(\tau) - q^{-\frac{n+1}{2}} d f_{n-1}(\tau) = q^{\frac{n}{2}} f_n(\tau') - q^{\frac{n+1}{2}} f_{n-1}(\tau') d, \tag{2.2}$$

$$d_n = q^{\frac{n}{2}} f_n(\tau) - q^{\frac{n+1}{2}} a f_{n-1}(\tau) = q^{-\frac{n}{2}} f_n(\tau') - q^{-\frac{n+1}{2}} f_{n-1}(\tau') a. \tag{2.3}$$

Since these expressions (2.2), (2.3) (and (1.9), (1.10)) hold for $n = 0$, Corollary 1.2 is also true for the case of $n = 0$. (3) Umeda-Wakayama [UW] considered

$$\tau_n := \mathrm{tr}\, A^n C^n = q^{-\frac{n}{2}} a_n + q^{-\frac{n}{2}} b_n,$$

$$\tau_n' := \mathrm{tr}\, C^{-n} A^n = q^{-\frac{n}{2}} a_n + q^{\frac{n}{2}} b_n,$$

and pointed out that τ_n and τ_n' satisfy the following Fibonacci type equations:

$$\tau_{n+1} = \tau_n \tau - \tau_{n-1}\delta, \quad \tau'_{n+1} = \tau'_n \tau' - \tau'_{n-1}\delta. \tag{2.4}$$

These equations (2.4) are equal to the recursion (1.3) of $f_n(\tau)$ exactly. Hence by $\tau_1 = \tau$ and $\tau'_1 = \tau'$ we have

$$\tau_n = f_n(\tau), \quad \tau'_n = f_n(\tau'). \tag{2.5}$$

3 Applications

We point out that quantum adjoint matrix $\hat{A}$ is a q^{-1}-quantum matrix and

$$\hat\tau := q^{-\frac{1}{2}}\hat a + q^{\frac{1}{2}}\hat d = q^{\frac{1}{2}} a + q^{-\frac{1}{2}} d = \tau, \quad \hat\tau' := q^{\frac{1}{2}}\hat a + q^{-\frac{1}{2}}\hat d = q^{-\frac{1}{2}} a + q^{\frac{1}{2}} d = \tau', \quad \hat\delta := \hat a\hat d - q^{-1}\hat b\hat c = da - q^{-1}bc = \delta.$$

Then we prove Corollary 1.3.

From (1.5), Corollary 1.2 and Corollary 1.3, we obtain the proof of Theorem 1.4 which is a simple prood of Vokos-Zumino-Wess [VZW].

Proof of Theorem 1.4 For any non-negative integers m,n, from Corollary 1.3 we have

$$
\begin{aligned}
\hat{A}^m A^n &= \begin{pmatrix} d_m & -q^{-m} b_m \\ -q^m c_m & a_m \end{pmatrix} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \\
&= \begin{pmatrix} d_m a_n - q^{-m} b_m c_n & d_m b_n - q^{-m} b_m d_n \\ -q^m c_m a_n + a_m c_n & -q^m c_m b_n + a_m d_n \end{pmatrix}
\end{aligned} \tag{3.1}
$$

and

$$
\begin{aligned}
A^n \hat{A}^m &= \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} d_m & -q^{-m} b_m \\ -q^m c_m & a_m \end{pmatrix} \\
&= \begin{pmatrix} a_n d_m - q^m b_n c_m & -q^{-m} a_n b_m + b_n a_m \\ c_n d_m - q^m d_n c_m & -q^{-m} c_n b_m + d_n a_m \end{pmatrix}.
\end{aligned} \tag{3.2}
$$

On the other hand, by applying $A\hat{A} = A\hat{A} = \delta E_2$ we obtain

$\hat{A}^m A^n = A^n \hat{A}^m$

$\delta^m A^{n-m} = \delta^m c_{n-m} \ldots \delta\delta^{m-m} d b_{n-m} \ldots$

$(m < n)$ $\delta^m a$

$=$.

$\hat{A}^{m-n} = -q^m \delta^n d_{n-m} \delta^n c_{n-m} - q^n \delta^n a_{m-n} \delta^n b_{m-n}$ $(m \geq n)$ δ

(3.3)

By comparing the entries of (3.1), (3.2) and (3.3) we have (1.16), (1.17), (1.18) and (1.19).

Finally, the relation (1.20) follows from the explicit formulas (1.9) and (1.10):

$$
\begin{aligned}
b_n c_m &= q^{\frac{n-1}{2}} f_{n-1}(\tau') b q^{-\frac{m-1}{2}} c f_{m-1}(\tau) \\
&= q^{n-m} q^{-\frac{n-1}{2}} f_{n-1}(\tau') c q^{\frac{m-1}{2}} b f_{m-1}(\tau) \\
&= q^{n-m} c_n b_m.
\end{aligned}
$$

□

If we set $m = n$ in (1.16), (1.17), (1.18), (1.19) and (1.20), then we obtain an interesting Corollary which means that A^n is a 2×2 q^n-quantum matrix.

Corollary 3.1. For any non-negative integer n, we have $a_n b_n = q_n b_n a_n$, $a_n c_n = q_n c_n a_n$, $a_n d_n - d_n a_n = (q_n - q_{-n}) b_n c_n$,

$b_n c_n = c_n b_n$, $b_n d_n = q_n d_n b_n$, $c_n d_n = q_n d_n c_n$ and $\quad (R_{q_n})$

(3.4)

$$a_n d_n - q_n b_n c_n = d_n a_n - q_{-n} b_n c_n = \delta_n. \text{ i.e.}$$

(the quantum determinant of A^n) = (the quantum determinant of A)^n

Originally, Theorem 1.4 was proved by Vokos-Zumino-Wess [VZW] and its proof was a brute force approach using double induction on m and n. Later, Corrigan-Fairlie-Fletcher-Sasaki [CFFS] and Umeda-Wakayama [UW] gave some simple proofs of Corollary 3.1 which is the case of $m = n$ in Theorem 1.4 independently. Our proof of Theorem 1.4 is different from any of them.

In particular, it is desirable to extend Theorem 1.1 and Corollary 1.2 to $n \times n$ quantum matrices.

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